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Correlation Inequalities for Antiferromagnets

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We prove several correlation inequalities for a class of antiferromagnets with pair interaction and arbitrary fields. To do this we use the Ginibre–Percus method of double variables.

KEY WORDS: Correlation inequalities; antiferromagnetic systems; arbitrary fields; Ginibre–Percus method.

1. INTRODUCTION

Correlation inequalities play an important role in rigorous statistical mechanics. Therefore it would be of some interest to be able to find more of them. Almost all known inequalities are valid only in the ferromagnetic region, the only exception being the FKG inequalities, via the isomorphism between ferromagnets with a staggered external field and antiferromagnets with a constant external field.

In this paper we describe some new inequalities of the Griffiths type for lattice antiferromagnetic systems with arbitrary spins. These inequalities will be derived directly for antiferromagnetic systems. They also yield by judicious flippings of spins on suitable sublattices some new inequalities for ferromagnetic systems. In some cases, however, e.g., in some examples described below, they are consequences of the FKG inequalities.

The following is the simplest example of our results. Consider an antiferromagnet with pair translation-invariant interaction and constant magnetic field. Also choose some "chessboard" configuration, which is equal to any real constant on "white squares" and to any smaller constant on "black squares." This staggered configuration is defined on the whole lattice and plays the role of the boundary conditions. Then in the infinite-volume limit the expectation of the difference between any "white" spin and "black"

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spin is positive. An interesting point here is that the result, contrary to the usual feeling, fails in general for a finite volume.

This inequality is a by-product of the attempt to prove the following conjecture: Consider the above-mentioned expectation of the difference between "white-square" spin and "black-square" spin as a function of external field. It increases monotonically on some half-line and decreases monotonically on its complement (at least for spin 1/2).

The GHS inequality indicates the conjecture to be true in some neighborhood near the maximum of our function if the interaction is of a special form and the volume is finite. However, this neighborhood depends on the size of the volume and goes to zero as the volume goes to infinity.

2. NOTATIONS AND RESULTS

Let $\mathbb{Z}^\nu$ be a ν-dimensional cubic lattice and $L_e \subset \mathbb{Z}^\nu$ be any sublattice of index two; $L_o = \mathbb{Z}^\nu \backslash L_e$ is the coset (here e denotes even and o odd). For $i \in \mathbb{Z}^\nu$ let

$$p(i) = \begin{cases} 0, & i \in L_e, \\ 1, & i \in L_o \end{cases}$$

(the parity of i), and let $p(A) = \sum_{i \in A} p(i)$ for $A \subset \mathbb{Z}^\nu$ and finite.

For any $i \in \mathbb{Z}^\nu$ let $\sigma_i \in \mathbb{R}^1$ denote the value of the spin situated at point i. The joint distribution of the variables $\sigma_\Lambda \equiv \{\sigma_i; i \in \Lambda, \Lambda \subset \mathbb{Z}^\nu \text{ and finite}\}$ is given by the measure

$$dP_{\Lambda,\bar{\sigma}} = Z^{-1}(\Lambda, \bar{\sigma}) \exp\{-H_\Lambda(\sigma_\Lambda | \bar{\sigma})\}\, d\mu_\Lambda \tag{1}$$

where $\bar{\sigma}: \mathbb{Z}^\nu \to \mathbf{A}^1$ is some function or configuration, which is called the *boundary condition*; $\mu_\Lambda = \bigoplus_{I \in \Lambda} \mu_i$, $\mu_i \equiv \mu$, μ is any symmetric measure on $\mathbb{R}^1$; the *Hamiltonian* H_Λ is given by

$$-H_\Lambda(\sigma_\Lambda | \bar{\sigma}) = \tfrac{1}{2} \sum_{i,j \in \Lambda} J_{ij} \sigma_i \sigma_j + \sum_{i \in \Lambda} h_i \sigma_i + \sum_{i \in \Lambda, j \notin \Lambda} J_{ij} \sigma_i \bar{\sigma}_j \tag{2}$$

and the factor Z^{-1} normalizes the measure (1). [In the following we suppose the convergence of everything we need (this is a condition on μ). In particular, the factor $Z(\Lambda, \bar{\sigma})$ exists.]

The interaction J_{ij} is supposed to be antiferromagnetic, namely

$$(-1)^{p(\{i,j\})} J_{ij} \geqslant 0 \tag{3}$$

Now, let $(\mathbb{Z}^\nu)'$ be an isomorphic copy of our lattice. We shall use the same letters for corresponding points in $\mathbb{Z}^\nu$ and $(\mathbb{Z}^\nu)'$; other objects on $(\mathbb{Z}^\nu)'$ will be denoted by a prime. So we have some antiferromagnetic interaction J'_{ij}, external field h_i', and boundary condition $\bar{\sigma}'$; the measure μ' is equal to μ.

To state our first result let $\tau \in L_o'$ and consider the family $\sigma'_{\Lambda'+\tau} = \{\sigma_i', i \in \Lambda' + \tau\}$. Then we have the following.

Theorem 1. If

$$1. \qquad J'_{i+\tau,j+\tau} = J_{ij}$$

$$2. \qquad (-1)^{p(i)}(h_i - h'_{i+\tau}) \geqslant 0$$

$$3. \qquad (-1)^{p(i)}(\bar{\sigma}_i - \bar{\sigma}'_{i+\tau}) \geqslant 0$$

then

$$(-1)^{p(A)}\left\langle \prod_{i \in A \subset \Lambda} (\sigma_i - \sigma'_{i+\tau}) \right\rangle_{\Lambda,\Lambda'+\tau} \geqslant 0$$

Here $\langle \cdots \rangle_{\Lambda,\Lambda'+\tau}$ means integration with respect to the measure

$$P_{\Lambda,\sigma} \otimes P'_{\Lambda'+\tau,\sigma'}$$

Note. This theorem is similar to the Percus inequality; see Sylvester.[1]

To obtain the corresponding theorem for ferromagnetic systems, we flip all spins in the sublattices L_o and L_e. Condition (2), e.g., becomes

$$(h_i - h_i{}'_{i+\tau}) \geqslant 0$$

It is immediate that we can replace T_i in the conclusions of Theorem 1 by any monotone function F_i of T_i. If $A = di\xi$ is a one-point set, then we get a special case of the FKG Holley inequality. If $A = \{i, j\}$, then the corresponding inequality can also be derived by applying the FKG Holley inequality twice. Furthermore, if the functions F_i are positive, we get (see proof of Corollary 2)

$$\left\langle \prod_{i \in A \subset \Lambda} F_i(\sigma_i) - \prod_{i \in A \subset \Lambda} F_i(\sigma_{i+\tau}) \right\rangle_{\Lambda,\Lambda'+\tau} \geqslant 0$$

which also is a special case of the FKG Holley inequality.

In the following corollary we need the existence of the infinite-volume limit and its independence from the sequence of finite boxes. For the case $\sigma_i = \pm 1$ it is an easy consequence of the FKG inequalities, as was found by Lebowitz and Martin-Löf.[2] Their method also covers the case of measures μ with compact support. However, the case of arbitrary μ is not so simple, and Lebowitz and Presutti[3] found sufficient conditions on μ and the J_{ij} under which the desired limit exists. In what follows, we simply suppose this.

Corollary 1. In addition to Theorem 1, suppose

$$J_{ij} = J(i - j)$$

$$h_i = \begin{cases} h_e, & i \in L_e \\ h_o, & i \in L_o \quad \text{and} \quad h_e \geqslant h_o \end{cases} \tag{4}$$

$$\bar{\sigma}_i = \begin{cases} \bar{\sigma}_e, & i \in L_e \\ \bar{\sigma}_o, & i \in L_o \quad \text{and} \quad \bar{\sigma}_e \geqslant \bar{\sigma}_o \end{cases} \tag{5}$$

Let $\langle\cdots\rangle$ be the limiting measure $P_{\Lambda,\sigma}$ as $\Lambda \to \mathbb{Z}^\nu$. Then

$$\langle\sigma_i\rangle \geqslant \langle\sigma_j\rangle \qquad \text{for} \quad i \in L_e, \quad j \in L_o$$

Corollary 2. If, in addition, supp $\mu \in [-1, 1]$, then

$$\langle\sigma_i\rangle - \langle\sigma_j\rangle \geqslant (1/\sqrt{2}l)|\langle\sigma_i^2\rangle - \langle\sigma_j^2\rangle| \tag{6}$$

Note. The idea of (6) is similar to that of Lebowitz.[4],[2]

Now let $P \subset \mathbf{A}^\nu$ be a hyperplane, and let a tilde superscript denote the corresponding reflection. Suppose that $(\mathbb{Z}^\nu)^\sim = \mathbb{Z}^\nu$, $L_e^\sim = L_o$, and $\Lambda^\sim = \Lambda$. Then we have:

Theorem 2. Let the antiferromagnetic interaction J_{ij} be rotation-invariant $[J_{ij} = J(|i - j|)]$, $h_i \geqslant h_{\tilde{i}}$, $\bar{\sigma}_i \geqslant \bar{\sigma}_{\tilde{i}}$, for any $i \in L_e$. Then

$$\left\langle \prod_{i \in A \subset \Lambda \cap L_e} (\sigma_i - \sigma_{\tilde{i}}) \right\rangle_\Lambda \geqslant 0$$

Corollary 3. Let (4) and (5) hold. Then in the infinite-volume limit

$$\langle\sigma_i\sigma_j\rangle + \langle\sigma_{i+\lambda}\sigma_{j+\lambda}\rangle \geqslant 2\langle\sigma_s\sigma_t\rangle$$

where $i, j, s \in L_e$; $\lambda, t \in L_o$; and vectors $j - i$, $s - j$, $t - s$, and λ are parallel to some unit vector from $\mathbb{Z}^\nu$.

Note. The last two statements result from the application of Hegerfeld's method[5] in our case.

3. PROOFS

The proofs follow the general approach of Ref. 6.

Proof of Theorem 1. The joint distribution of the family $\{\sigma_\Lambda, \sigma'_{\Lambda'+\tau}\}$ is given by the product measure:

$$Z^{-1}(\Lambda, \bar{\sigma})Z^{-1}(\Lambda' + \tau, \bar{\sigma}') \exp\{-H_\Lambda(\sigma_\Lambda|\bar{\sigma}) - H_{\Lambda'+\tau}(\sigma'_{\Lambda'+\tau}|\bar{\sigma}')\}\, d\mu_\Lambda\, d\mu_{\Lambda'+\tau} \tag{7}$$

[2] Similar inequalities were proven by Lebowitz.[7]

Let

$$x_i = (\sigma_i + \sigma'_{i+\tau})/\sqrt{2}, \qquad \bar{x}_i = (\bar{\sigma}_i + \bar{\sigma}'_{i+\tau})/\sqrt{2}$$

$$y_i = (-1)^{p(i)}(\sigma_i - \sigma'_{i+\tau})/\sqrt{2}, \qquad \bar{y}_i = (-1)^{p(i)}(\bar{\sigma}_i - \bar{\sigma}_{i+\tau})/\sqrt{2}$$

$$h_i^+ = (h_i + h'_{i+\tau})/\sqrt{2}, \qquad h_i^- = (-1)^{p(i)}(h_i - h'_{i+\tau})/\sqrt{2}$$

We shall make use of the following identities:

$$\begin{aligned} uv + u'v' &= \tfrac{1}{2}[(u + u')(v + v') + (u - u')(v - v')] \\ &= \tfrac{1}{2}[(u + u')(v + v') + (u' - u)(v' - v)] \\ &= \tfrac{1}{2}[(u + u')(v + v') - (u - u')(v' - v)] \\ &= \tfrac{1}{2}[(u + u')(v + v') - (u' - u)(v - v')] \end{aligned}$$

The first pair appears in every paper on this subject. Using the second pair of identities, we shall be able to eliminate some of the antiferromagnetic negativeness in the Hamiltonian. Namely, we have

$$\begin{aligned} &-H_\Lambda(\sigma_\Lambda|\bar{\sigma}) - H_{\Lambda'+\tau}(\sigma'_{\Lambda'+\tau}|\bar{\sigma}') \\ &\quad\equiv -H(x_\Lambda, y_\Lambda|\bar{x}, \bar{y}) \\ &\quad= \tfrac{1}{2}\sum_{i,j\in\Lambda} J_{ij}x_ix_j + \sum_{i\in\Lambda, j\notin\Lambda} J_{ij}x_i\bar{x}_j + \sum_{i\in\Lambda} h_i^+ x_i \\ &\qquad + \tfrac{1}{2}\sum_{i,j\in\Lambda} |J_{ij}|y_iy_j + \sum_{i\in\Lambda, j\notin\Lambda} |J_{ij}|y_i\bar{y}_j + \sum_{i\in\Lambda} h_i^- y_i \end{aligned}$$

Let $H^+(x_\Lambda|\bar{x})$ be the first half of the last sum and $H^-(y_\Lambda|\bar{y})$ be the second half. The inequality of Theorem 1 can be rewritten in the form

$$\int_{\mathbb{R}^{2|\Lambda|}} \prod_{i\in A} y_i \exp\{-H(x_\Lambda, y_\Lambda|\bar{x}, \bar{y})\}\, d\mu_\Lambda(\sigma_\Lambda)\, d\mu_{\Lambda'+\tau}(\sigma_{\Lambda'+\tau}) \geqslant 0 \tag{8}$$

Now, $H^-(y_\Lambda|\bar{y})$ is a polynomial in y_i, $i \in \Lambda$, with nonnegative coefficients, according to the hypothesis of the theorem. By expanding the exponent $\exp\{-H^-(y_\Lambda|\bar{y})\}$, we find that the left-hand side of (8) becomes a sum of a series, with each term given, up to a nonnegative factor, by the integral

$$\int_{\mathbb{R}^{2|\Lambda|}} \prod_{i\in\Lambda} y_i^{n(i)} \exp\{-H^+(x_\Lambda|\bar{x})\}\, d\mu_\Lambda(\sigma_a)\, d\mu_{\Lambda'+\tau}(\sigma'_{\Lambda'+\tau}), \qquad n(i) \in \mathbb{Z}^+ \tag{9}$$

But the transformation

$$\sigma_i \to \sigma'_{i+\tau}, \qquad \sigma'_{i+\tau} \to \sigma_i \tag{10}$$

leaves unchanged the measure $\mu_\Lambda \otimes \mu_{\Lambda'+\tau}$ and the function $H^+(x_\Lambda|\bar{x})$, changing the sign of y_i. Thus, the integral (9) is equal to zero if $n(i)$ is odd for some $i \in \Lambda$; otherwise the integrand is nonnegative. QED

Corollary 1 follows by observing that $\lim_{\Lambda\to\infty} P_{\Lambda,\bar{\sigma}} = \lim_{\Lambda\to\infty} P_{\Lambda+\tau,\sigma}$ (see Refs. 2 and 3).

To prove Corollary 2, consider any function $g(\sigma_\Lambda, \sigma_{\Lambda'+\tau})$, which is nonnegative $(\mu_\Lambda \otimes \mu_{\Lambda'+\tau})$-almost everywhere and invariant under transformation (10). It is easy to see that the integral (9) remains nonnegative after multiplying the integrand by g. In particular, if $\operatorname{supp}\mu \subset [-l, l]$, then the functions $(\sqrt{2}l \pm x_i)$ can be taken for g.

Proof of Theorem 2. Let

$$x_i = (\sigma_i + \sigma_{\bar{i}})/\sqrt{2}, \qquad y_i = (\sigma_i - \sigma_{\bar{i}})/\sqrt{2}, \qquad i \in \Lambda \bigcap L_e$$

$$\bar{x}_i = (\bar{\sigma}_i + \bar{\sigma}_{\bar{i}})/\sqrt{2}, \qquad \bar{y}_i = (\bar{\sigma}_i - \bar{\sigma}_{\bar{i}})/\sqrt{2}$$

$$h_i^+ = (h_i + h_{\bar{i}})/\sqrt{2}, \quad h_i^- = (h_i - h_{\bar{i}})/\sqrt{2}, \qquad i \in L_e$$

For any $s, t \in \Lambda$, $s \neq t \neq \bar{s}$, let $\{i, j\}$ be the intersection $\{s, t, \bar{s}, \bar{t}\} \bigcap L_e$. Then

$$J_{st}\sigma_s\sigma_t + J_{s\bar{t}}\sigma_s\sigma_{\bar{t}} + J_{\bar{s}t}\sigma_{\bar{s}}\sigma_t + J_{\bar{s}\bar{t}}\sigma_{\bar{s}}\sigma_{\bar{t}} = (J_{ij} + J_{i\bar{j}})x_i x_j + (J_{ij} - J_{i\bar{j}})y_i y_j$$

because $J_{ij} = J_{\bar{i},\bar{j}}$, $J_{i\bar{j}} = J_{\bar{i}j}$. Hence

$$\begin{aligned}
-H_\Lambda(\sigma_\Lambda|\bar{\sigma}) &= \tfrac{1}{2} \sum_{i,j\in\Lambda\cap L_e} (J_{ij} + J_{i\bar{j}})x_i x_j + \sum_{i\in\Lambda\cap L_e, j\in\bar{\Lambda}\cap L_e} (J_{ij} + J_{i\bar{j}})x_i\bar{x}_j \\
&+ \sum_{i\in\Lambda\cap L_e} h_i^+ x_i + \tfrac{1}{2} \sum_{i,j\in\Lambda\cap L_e} (J_{ij} - J_{i\bar{j}})y_i y_j \\
&+ \sum_{i\in\Lambda\cap L_e, j\in\Lambda\cap L_e} (J_{ij} - J_{i\bar{j}})y_i\bar{y}_j + \sum_{i\in\Lambda\cap L_e} h_i^- y_i + \sum_{i\in\Lambda\cap L_e} {}_{i\bar{i}}\sigma_i\sigma_{\bar{i}}
\end{aligned}$$

By definition, $J_{ij} - J_{i\bar{j}} \geqslant 0$, provided $i, j \in L_e$. The remaining part of the proof coincides with that of Theorem 1.

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